2. $||\lambda x|| = ||\lambda|||x||$ $(x \in E, \lambda \in \mathbf{C}$ *or* $\mathbf{R}$*resp.*);

3. $||x|| = 0$ *implies* $x = 0$ $(x \in E)$.

A **normed space** *is a pair* $(E, ||||)$ *where E is a vector space and* $||||$ *is a norm on E.*

If $||||$ is a seminorm (resp. a norm) the mapping

$$d_{||||} : (x,y) \to ||x - y||$$

is a semimetric (resp. a metric) on E. We call it the **semimetric** (resp. metric) **induced by** $||||$. Thus every normed space $(E||||)$ can be regarded in a natural way as a metric space and so as a topological space and we can use, in the context of normed spaces, such notions as continuity of mappings, convergence of sequences or nets, compactness of subsets etc.

If $(E, ||||)$ is a normed spaces, we write $B_{||||}$ or $B(E)$ for the closed unit ball of E i.e. the set $\{x \in E : ||x|| \leq 1\}$.

Exercises.

A. A subset A of a vector space is **absolutely convex** if $\lambda x + \mu y \in A$

whenever $x,y \in A$, $\lambda,\mu \in \mathbf{C}$ (respectively $\mathbf{R}$) and $|\lambda| + |\mu| \leq 1$. $A\lambda| \leq$isis

absolutely convex and absorbing then**absorbing**ρ. Show thatif for

each$B_{||||}$ is absolutely convex and absorbing and that if$x \in E$ there is a ρ

> 0 so that $\lambda x \in A$ when

|
A

$$||||_A : x \mapsto \inf\{\rho > 0 : x \in \rho A\}$$

is a seminorm on E (it is called the **Minkowski functional** of A).

useful on occasions when a natural construction "should" produce a normed space but in fact only produces a seminormed space. We simply factor out the zero subspace.)

As is customary in mathematics, we identify spaces which have the same structure. The appropriate concept is that of an isomorphism. It turns out that there are two natural ones in the context of normed spaces:

Let E,F be normed spaces. E and F are **isomorphic** if there is a bijective linear mapping $T : E \to F$ so that**isomorphism**T is a homeomorphism or the.

If T is, in addition, norm topologies. T is then called an norm-preserving (i.e. $||Tx|| = ||x||$ for $x \in E$), T is an **isometry** and E and F are **isometrically isomorphic** (we write $E \sim F$ resp. $E \sim= F$ to indicate that E and F are isomorphic— resp. isometrically isomorphic).

Two norms $||||$ and $||||_1$ on a vector space E are **equivalent** if Id_E is an isomorphism from $(E,||||)$ onto $(E,||||_1)$ i.e. if $||||$ and $||||_1$ induce the same topology on E.

Isomorphisms are characterised by the existence of estimates from above and below—let $T : E \to F$ be a bijective linear mapping. Then T is an isomorphism if and only if there exist M and m (both positive) so that

$$m||x|| \leq ||Tx|| \leq M||x|| \qquad (x \in E)$$

(for a proof see the Exercise below).

Thus the norms $||||$ and $||||_1$ on E are equivalent if and only if there are $M,m > 0$ so that $m||x|| \leq ||x_1|| \leq M||x||$ $(x \in E)$.

We now bring a list of some simple examples of normed spaces. in the course of the later chapters we shall extend it considerably.

Examples

A. The following mappings on $\mathbf{C}^n$ (resp. $\mathbf{R}^n$) are norms:

$$\begin{aligned} || \, ||_1 &: (\xi_1, \ldots, \xi)n) \mapsto (|\xi_1| + \cdots + |\xi_n|) \\ || \, ||_2 &: (\xi_1, \ldots, \xi)n) \mapsto (|\xi_1|^2 + \cdots + |\xi_n|^2)^{1/2} \\ || \, ||_3 &: (\xi_1, \ldots, \xi)n) \mapsto \sup(|\xi_i| : i = 1, \ldots, n\}. \end{aligned}$$

3 Banach spaces

As mentioned in the introduction, we require a suitable completeness condition on our normed space in order to obtain more substantial results. The natural one is that of Cauchy comleteness with respect to the induced metric.

Definition 6 *A normed space* $(E, ||\,||)$ *is called a* **Banach space** *if it is complete under the associated metric i.e. if each Cauchy sequence converges.*

A useful property of Banach spaces is the following: if (x_n) *is a sequence so that* $P||x_n|| < \infty$*, then* Px_n *converges i.e. the partial sums* $\sum_{k=1}^{n} x_k$ *converge to a*

1. Show that the topology induced bystriction to G of the topology of E; $||||_G$ on G coincides with the re-

2. if $x \in xE$to, show that$G$ i.e. $||\pi_G(x)||$ (the norm in E/G) is just the distance from $\inf\{||x - y|| : y \in G\}$.

   Deduce that the seminorm on E/G is a norm if and only if G is closed. Use this to give an example where it is not a norm.

3. Show that the topology induced by the norm on $E \times F$ is the product of the topologies on E and F.

It follows from the very definition of the topology via the norm that it is very closely related to the linear structure of E. In fact, the following properties are valid:

1. the mappings

$$A : (x,y) 7\to x + y$$

   and

$$M : (\lambda,x) 7\to \lambda x$$

   from $E \times E$ into E resp. $\mathbf{C} \times E$ or $\mathbf{R} \times E$ into E are continuous for the topology generated by the norms. For

$$||(x,y) - (x_1 + y_1)|| \leq ||x - x_1|| + ||y - y_1||$$

   and

$$\begin{aligned} ||\lambda x - \lambda_1 x_1|| &= ||(\lambda x - \lambda_1 x) + (\lambda_1 x - \lambda_1 x_1)|| \\ &\leq |\lambda - \lambda_1| \, ||x|| + |\lambda_1| \, ||x - x_1||. \end{aligned}$$

2. Let G be a subspace of $(E,||||)$. Then the closure $\bar{G}$ of G is also a subspace.

Exercises. Let $||||$ be a seminorm on E. Show that $E_0 := \{x \in E : ||x|| = 0\}$ is a subspace of, show that E. If π_0 denotes the natural projection from E onto E/E_0 $\pi_0(x) \to ||x||$

is a well-defined mapping on E/E_0 and is, in fact, a norm. E/E_0, with this norm, is called the **normed space associated** with E. (This simple exercise is often